Phys 410
Fall 2015
Lecture #16 Summary
22 October, 2015

We considered the two-body problem of two objects interacting by means of a conservative central force, with no other external forces acting. This problem eventually simplifies from that of 6 degrees of freedom (for 2 particles in three dimensions) to essentially a single particle moving in one dimension! The Lagrangian can be simplified by adopting the generalized coordinates: relative coordinate $\vec{r} = \vec{r}_1 - \vec{r}_2$, and the center of mass coordinate $\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/M$, where $M = m_1 + m_2$ is the total mass. The two-particle Lagrangian simplifies to $\mathcal{L}(\vec{R}, \vec{r}) = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$, where $\mu = m_1 m_2/M$ is called the reduced mass because it is smaller than either m_1 or m_2. Because this Lagrangian is independent of $\vec{R}$, it means that the center of mass (CM) momentum $M\dot{\vec{R}}$ is constant. The other Lagrange equation gives $\mu\ddot{\vec{r}} = -\vec{\nabla}U(r)$, which is Newton's second law for the relative coordinate.

Taking advantage of the CM conserved momentum, we can jump to the CM (inertial) reference frame, where the CM is at rest, and the two particles are always moving with equal and opposite momenta (this follows from the fact that $\dot{\vec{R}} = 0$ in the CM reference frame). In this reference frame, the Lagrangian simplifies to $\mathcal{L} = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$. Because only central forces act, the net torque that the particles exert on each other is zero, hence the total angular momentum of the particles ($\vec{L}$) as seen in this reference frame is conserved. Writing the sum of the angular momenta of the two particles, as seen in the CM reference frame, in terms of the generalized coordinates, we find $\vec{L} = \vec{r} \times \mu\dot{\vec{r}}$, which is the same as the angular momentum of a single particle of mass μ. Because $\vec{L}$ is conserved (including its direction), the vectors $\vec{r}$ and $\dot{\vec{r}}$ must remain in a fixed two-dimensional plane throughout the motion. This means that the motion is strictly two-dimensional! Note that purely 2D motion (and the concept of a trajectory) is prohibited in quantum mechanics, hence the reduced mass particle in the hydrogen atom problem spreads into a "cloud" of probability density, very roughly speaking.

Now we have to solve the remaining two-dimensional motion problem with this Lagrangian: $\mathcal{L} = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$. Going over to polar coordinates for $\vec{r}$, we get $\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r)$, as derived earlier. There are two Lagrange equations that follow from this Lagrangian. First we note that φ is an ignorable coordinate, hence the angular momentum of the fictitious 'particle' is conserved: $\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \mu r^2\dot{\varphi} = constant$. This is in fact just the z-component of the total angular momentum vector $\vec{L}$ that we calculated above. We give it a new name, ℓ, because it is a constant of the motion (you may now recognize the notation from the quantum treatment of the Hydrogen atom). The other Lagrange equation (for r) gives $\mu\ddot{r} = \mu r\dot{\varphi}^2 - dU/dr$. The first term on the RHS is the centrifugal force for the 'particle'.

The two Euler-Lagrange equations can be combined into a single equation for the relative coordinate: $\mu\ddot{r} = \ell^2/\mu r^3 - dU/dr$. This can be expressed in terms of the effective potential as $\mu\ddot{r} = -dU_{eff}/dr$, where $U_{eff}(r) = U(r) + \ell^2/2\mu r^2$.

Using the example of gravity for $U(r)$ we found that the effective potential (for $\ell > 0$) has a minimum at a finite value of r, diverges as r goes to zero, and approaches zero from below as r goes to infinity. We found that mechanical energy for the relative coordinate is conserved and equal to $E = \frac{\mu\dot{r}^2}{2} + \frac{\ell^2}{2\mu r^2} + U(r)$. Since kinetic energy is either positive or zero, the particle must be located in a region where $E \geq U_{eff,min}$. We see that when $E > 0$ the particle has an un-bounded orbit, while when $E < 0$ it has a bounded orbit trapped between minimum and maximum values of r.

We then solved the radial equation $\mu\ddot{r} = \frac{\ell^2}{\mu r^3} + F(r)$ for inverse-square force-laws of the form $F(r) = -\gamma/r^2$, and found a solution that expressed the radial coordinate in terms of the angular polar coordinate, $r = r(\varphi)$, in which time has been eliminated. The result is $r(\varphi) = \frac{c}{1+\epsilon\cos\varphi}$, where $c = \frac{\ell^2}{\mu\gamma}$ is a length scale and ϵ is an un-determined constant. This is the equation for the orbit of a planet around the sun, or a satellite around the earth.